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Continuous subgroups of the Poincaré group P(1, 4)

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Abstract. An exhaustive description of the non-splitting subalgebras of the LP(1, 4) algebra with respect to P(1, 4) conjugation is presented.

1. Introduction

The generalised Poincaré group P(1, 4) is the group of inhomogeneous pseudo-orthogonal transformations of the five-dimensional pseudo-Euclidean space with the scalar product $(X, Y) = x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 - x_4y_4$. The P(1, 4) group is the simplest one which contains the Poincaré group P(1, 3) as a subgroup. Fushchich and Krivsky (1968, 1969) and Fuschchich (1970) have used the P(1, 4) group and its unitary representations to describe particles with variable mass and spin. An arbitrary partial differential equation which is invariant under the P(1, 4) group is also invariant under the P(1, 3) group as well as under the extended Galilei group $\tilde{G}(1, 3)$ since $\tilde{G}(1, 3) \subset$ P(1, 4) (Fushchich and Nikitin 1980). The papers of Aghassi *et al* (1970a, b) deal with irreducible representations of P(1, 4) and G(1, 4), using the latter in the theory of elementary particles. Kadyshevsky (1980) proposed using the P(1, 4) group in field theory with the fundamental length. The P(1, 4) group is the invariance group of the relativistic Hamilton-Jacobi equation (Fuschchich and Serov 1983a) and the Monge-Ampere equation (Fushchich and Serov 1983b). These nonlinear equations are invariant under transformations of the P(1, 4) group with the fifth coordinate as $x_4 \equiv u$, where $u = u(x_0, x_1, x_2, x_3)$. So it is important to investigate the subgroup structure of the P(1, 4) group. In particular, these results can be used in the separation of variables of many important partial differential equations.

The splitting subalgebras of LP(1, 4) were described by Fedorchuk (1978, 1979). Some high-dimension non-splitting subalgebras of LP(1, 4) were listed by Fedorchuk and Fuschchich (1980) and Fedorchuk (1981). In this paper we list all the non-splitting subalgebras of the LP(1, 4) algebra with respect to P(1, 4) conjugation. In the papers of Lassner (1970), Bacry *et al* (1972, 1974a, b) and Patera *et al* (1975) all the subalgebras of LP(1, 3) are classified with respect to P(1, 3) conjugation, so we consider such subalgebras of LP(1, 4) which are non-conjugate to the subalgebras of LP(1, 3). In our paper we use the method due to Patera *et al* (1975).

2. Some auxiliary remarks

The LP(1, 4) algebra is defined by the following computation relations:

$$[J_{\alpha\beta}, J_{\gamma\delta}] = g_{\alpha\delta}J_{\beta\gamma} + g_{\beta\gamma}J_{\alpha\delta} - g_{\alpha\gamma}J_{\beta\delta} - g_{\beta\delta}J_{\alpha\gamma}$$

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$$[P_\alpha, J_{\beta\gamma}] = g_{\alpha\beta}P_\gamma - g_{\alpha\gamma}P_\beta \qquad J_{\beta\alpha} = -J_{\alpha\beta} \qquad [P_\alpha P_\beta] = 0$$

where $g_{00} = -g_{11} = -g_{22} = -g_{33} = -g_{44} = 1$, $g_{\alpha\beta} = 0$ if $\alpha \neq \beta$ $(\alpha, \beta = 0, 1, 2, 3, 4)$.

Below we shall use the following notation: $K_a = J_{0a} - J_{a4}$ $(a = 1, 2, 3)$; $W = \langle X_1, \ldots, X_s\rangle$ is a space or Lie algebra over the real number field R with the generating elements $X_1, \ldots, X_s$; $V = \langle P_0, P_1, P_2, P_3, P_4\rangle$; π is a projection LP(1, 4) on LO(1, 4); $\pi_{a,\ldots,q}$ is a projection LP(1, 4) on $\langle P_a, \ldots, P_q\rangle$.

Lemma 1. Let W be a subspace of V invariant under $\mathrm{Ad}J_{ab}$ $(1 \leq a < b \leq 4)$. If $\pi_{a,b}(W) \neq 0$ then $P_a, P_b \in W$.

Proof. Let $X = \sum x_\alpha P_\alpha \in W$ and $\pi_{a,b}(X) \neq 0$. Obviously,

$$[J_{ab}, X] = x_a P_b - x_b P_a \qquad [J_{ab}, [J_{ab}, X]] = -x_a P_a - x_b P_b.$$

Since the vectors obtained are linearly independent, so $P_a, P_b \in W$ and this proves the lemma.

Lemma 2. If $W \subset V$ and $[J_{0a}, W] \subset W$ and if $\pi_{0,a}(W) \neq 0$, then the subspace W contains $P_0 + P_a$ or $P_0 - P_a$.

Corollary. Let $W \subset V$ and $[J_{0a}, W] \subset W$. If $\pi_{0,a}(W) \neq 0$ then within the conjugation corresponding to the element

$$\mathrm{diag}(\underbrace{1, \ldots, -1}_{a+1}, \ldots, 1)$$

from O(1, 4) group W contains $P_0 + P_a$.

Lemma 3. Let W be a subspace of V invariant under $\mathrm{Ad}(J_{0a} + \gamma J_{cd})$ where $\gamma \in R$, $\gamma \neq 0$, 0, a, c, d are mutually different. Then $W = \pi_{0,a}(W) \oplus \pi_{c,d}(W) \oplus s\langle P_b\rangle$, where $s \in \{0, 1\}$, $b \notin \{0, a, c, d\}$.

Proof. If

$$X = \sum_0^4 \alpha_j P_j \in W$$

then W contains the elements

$$X_1 = [J_{0a} + \gamma J_{cd}, X] = -\alpha_0 P_a - \alpha_a P_0 + \gamma(\alpha_c P_d - \alpha_d P_c)$$

$$X_2 = [J_{0a} + \gamma J_{cd}, X_1] = \alpha_0 P_0 + \alpha_a P_a + \gamma^2(-\alpha_c P_c - \alpha_d P_d)$$

$$X_3 = [J_{0a} + \gamma_{cd}, X_2] = -\alpha_0 P_a - \alpha_a P_0 + \gamma^3(-\alpha_c P_d + \alpha_d P_c).$$

Since $X_1 - X_3 = (\gamma + \gamma^3)(\alpha_c P_d - \alpha_d P_c)$ and $\gamma \neq 0$, then $\alpha_c P_d - \alpha_d P_c \in W$ whence $\pi_{c,d}(X)$, $\pi_{0,a}(X) \in W$. Thus, this lemma is proved.

Lemma 4. Let W be a subspace of V invariant under $\mathrm{Ad}K_a$. If $\pi_{0,4}(W) \not\subset \langle P_0 + P_4\rangle$ then $P_0 + P_4$, $P_a \in W$. If $\pi_a(W) \neq 0$ then $P_0 + P_4 \in W$.

Proof. Let W contains the vector $X = \sum \alpha_j P_j$, then W also contains $X_1 = [X, K_a] = \alpha_a(P_0 + P_4) + (\alpha_0 - \alpha_4)P_a$, $X_2 = [X_1, K_a] = (\alpha_0 - \alpha_4)(P_0 + P_4)$. If $\alpha_0 - \alpha_4 \neq 0$ then $P_0 + P_4$, $P_a \in W$. If $\alpha_0 - \alpha_4 = 0$, $\alpha_a \neq 0$ then $P_0 + P_4 \in W$. Thus this lemma is proved.

Lemma 5. Let W be a subspace of V invariant under $\mathrm{Ad}(K_a - J_{bc})$, where $\{a, b, c\} = \{1, 2, 3\}$. Then W is invariant under $\mathrm{Ad}\, K_a$ and $\mathrm{Ad}\, J_{bc}$.

Proof. Let $X = K_a - J_{bc}$, $Y \in W$. Since $[X, [X, [X, Y]]] = [J_{bc}, Y]$, then $[J_{bc}, W] \subset W$, $[K_a, W] \subset W$. Thus, the lemma is proved.

Lemma 6. Let F be a subalgebra of LO(1, 4) with the generators J_{04} and K_a, where a covers a subset I of the set $\{1, 2, 3\}$. If A is a subalgebra of LP(1, 4) and $\pi(\mathrm{A}) = \mathrm{F}$, then within the conjugation with respect to the group of translations A contains elements $K_a (a \in I)$ and $J_{04} + \delta_1 P_1 + \delta_2 P_2 + \delta_3 P_3$.

Proof. Let $X_a = K_a + \Sigma\, \alpha_i P_i$, $Y = J_{04} + \Sigma\, \delta_i P_i$ $(i = 0, 1, 2, 3, 4)$. By the automorphism $\exp(t_1 P_0 + t_2 P_4)$ the coefficients δ_0, δ_4 can be made zero. Since $[Y, X_a] = -K_a + \delta_a (P_0 + P_4) - \alpha_0 P_4 - \alpha_4 P_0$, one can therefore consider $X_a = K_a + \gamma P_0$ within the automorphism $\exp(tP_a)$. Evidently $[Y, X_a] + X_a = (\delta_a + \gamma) P_0 + (\delta_a - \gamma) P_4$. If $\gamma \neq 0$ then $P_0 + P_4 \in \mathrm{A}$ by lemma 4. Therefore we have P_0, $P_4 \in \mathrm{A}$ and hence $\gamma = 0$ within the conjugation. Thus, this lemma is proved.

Lemma 7. Let A be a subalgebra of LP(1, 4), $X = J_{12} + cJ_{04} + \beta P_3$, $Y = K_3 + \Sigma\, \gamma_i P_i$ $(i = 1, 2, 3, 4;\ c > 0)$. If X, $Y \in \mathrm{A}$, then A contains K_3.

Proof. It is easy to obtain

$$cY - [Y, X] = (\beta - c\gamma_4) P_0 + (c\gamma_1 - \gamma_2) P_1 + (c\gamma_2 + \gamma_1) P_2 + c\gamma_3 P_3 + (c\gamma_4 + \beta) P_4.$$

According to lemma 3 $(\beta - c\gamma_4) P_0 + (c\gamma_4 + \beta) P_4, (c\gamma_1 - \gamma_2) P_1 + (c\gamma_2 + \gamma_1) P_2 \in \mathrm{A}$. If $\gamma_4 \neq 0$ then lemma 4 yields P_0, $P_4 \in \mathrm{A}$. If $c\gamma_1 - \gamma_2 = 0$, $c\gamma_2 + \gamma_1 = 0$ then $\gamma_1 = \gamma_2 = 0$. Thereafter using lemma 1 we can put $\gamma_1 = \gamma_2 = 0$. Since $c\gamma_3 P_3 \in \mathrm{A}$ one can admit that $\gamma_3 = 0$. Thus the lemma is proved.

Lemma 8. Let A be a subalgebra of LP(1, 4), $\varphi = \exp(-\omega K_b)(\omega \in R, \omega \neq 0)$. If $P_0 + P_4$, $P_b + \omega^{-1} P_4 \in \mathrm{A}$ $(1 \leq b \leq 3)$ then the algebra $\varphi(\mathrm{A})$ contains P_0 and P_4.

Proof. According to the Campbell-Hausdorff formula we have

$$\varphi(P_0 + P_4) = P_0 + P_4 \qquad \varphi(P_b + \omega^{-1} P_4) = \omega^{-1} P_4 + \tfrac{1}{2}\omega (P_0 + P_4).$$

This gives that $P_0 + P_4$, $P_4 \in \varphi(\mathrm{A})$, therefore P_0, $P_4 \in \varphi(\mathrm{A})$. Thus this lemma is proved.

3. The non-splitting subalgebras of the LP(1, 4) algebra

Let $\tilde{\mathrm{F}}$ be an subalgebra of LP(1, 4) such that $\pi(\tilde{\mathrm{F}}) = \mathrm{F}$. An expression $\tilde{\mathrm{F}} + \mathrm{W}$ means that $[\mathrm{F}, \mathrm{W}] \subset \mathrm{W}$ and $\tilde{\mathrm{F}} \cap \mathrm{V} \subset \mathrm{W}$. As concerns the non-splitting algebras $\tilde{\mathrm{F}} + \mathrm{W}_1, \ldots, \tilde{\mathrm{F}} + \mathrm{W}_s$ we will use the notation $\tilde{\mathrm{F}}$: $\mathrm{W}_1, \ldots, \mathrm{W}_s$.

Theorem. Let α, β, δ, μ, $\omega \in R$, $\alpha > 0$, $\omega > 0$, $\mu \geq 0$ and this takes place for all labelling variables. The non-splitting subalgebras of the LP(1, 4) algebra are exhausted by the non-splitting subalgebras of the LP(1, 3) algebra and the following subalgebras:

$\langle J_{12} + \alpha P_0 \rangle$: $\langle P_3, P_4 \rangle$, $\langle P_1, P_2, P_3, P_4 \rangle$;

$\langle J_{12}+P_0+P_3\rangle$: $\langle P_4\rangle, \langle P_1, P_2, P_4\rangle$;

$\langle J_{12}+\alpha P_3\rangle$: $\langle P_4\rangle, \langle P_0+P_4\rangle, \langle P_0, P_4\rangle, \langle P_1, P_2, P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle$;

$\langle J_{12}+P_0\rangle$: $\langle P_0+P_4, P_3\rangle, \langle P_0+P_4, P_1, P_2, P_3\rangle$;

$\langle J_{12}+J_{34}+\alpha P_0\rangle$: $0, \langle P_1, P_2\rangle, \langle P_1, P_2, P_3, P_4\rangle$;

$\langle J_{12}+cJ_{34}+\alpha P_0\rangle$: $0, \langle P_1, P_2\rangle, \langle P_3, P_4\rangle, \langle P_1, P_2, P_3, P_4\rangle (0<c<1)$;

$\langle J_{04}+\alpha P_3\rangle$: $\langle P_1, P_2\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle$;

$\langle J_{12}+cJ_{04}+\alpha P_3\rangle$: $0, \langle P_0+P_2\rangle, \langle P_0, P_4\rangle, \langle P_1, P_2\rangle, \langle P_0+P_4, P_1, P_4\rangle, \langle P_0, P_1, P_2, P_4\rangle (c>0)$;

$\langle K_3+P_2\rangle$: $\langle P_1\rangle, \langle P_0+P_4, P_1\rangle, \langle P_0+P_4, P_1+\omega P_3\rangle, \langle P_0+P_4, P_1, P_3\rangle, \langle P_0, P_1, P_3, P_4\rangle$;

$\langle K_3+P_4\rangle$: $\langle P_1, P_2\rangle, \langle P_0+P_4, P_1+\omega P_3, P_2\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0+P_4, P_1, P_2, P_3\rangle$;

$\langle K_3-J_{12}+\alpha P_4\rangle$: $0, \langle P_0+P_4\rangle, \langle P_0+P_4, P_3\rangle, \langle P_1, P_2\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0+P_4, P_1, P_2, P_3\rangle$;

$\langle J_{12}+\alpha P_0, J_{34}+\mu P_0\rangle$: $0, \langle P_1, P_2\rangle, \langle P_1, P_2, P_3, P_4\rangle$; $\langle J_{12}, J_{34}+\alpha P_0, P_1, P_2\rangle$;

$\langle J_{04}+\alpha P_3, J_{12}+\mu P_3\rangle$: $0, \langle P_0+P_4\rangle, \langle P_0, P_4\rangle, \langle P_1, P_2\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle$;

$\langle J_{04}, J_{12}+\alpha P_3\rangle$: $0, \langle P_0+P_4\rangle, \langle P_0, P_4\rangle, \langle P_1, P_2\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle$;

$\langle J_{12}+P_0+P_4, K_3+\mu P_4\rangle$; $\langle J_{12}, K_3+P_4\rangle$;

$\langle J_{12}+\mu P_3, K_3+P_4, P_0+P_4\rangle$; $\langle J_{12}+\alpha P_3, K_3, P_0+P_4\rangle$;

$\langle J_{12}+P_0+P_4, K_3+\mu P_4, P_1, P_2\rangle$; $\langle J_{12}, K_3+P_4, P_1, P_2\rangle$;

$\langle J_{12}+\mu P_4, K_3+P_4, P_0+P_4, P_3\rangle$; $\langle J_{12}+P_4, K_3, P_0+P_4, P_3\rangle$;

$\langle J_{12}+\mu P_3, K_3+P_4, P_0+P_4, P_1, P_2\rangle$; $\langle J_{12}+\alpha P_3, K_3, P_0+P_4, P_1, P_2\rangle$;

$\langle J_{12}+\mu P_4, K_3+P_4, P_0+P_4, P_1, P_2, P_3\rangle$; $\langle J_{12}+P_4, K_3, P_0+P_4, P_1, P_2, P_3\rangle$;

$\langle K_1+\mu P_2+P_3, K_2+\mu P_1+\beta P_2\rangle$; $\langle K_1, K_2\pm P_2, P_3\rangle$;

$\langle K_1+P_2, K_2+P_1+\beta P_2, P_3\rangle$; $\langle K_1+\alpha P_2+P_3, K_2+\beta_1 P_1+\beta_2 P_2, P_0+P_4\rangle$;

$\langle K_1+P_3, K_2+\mu P_1+\beta P_2, P_0+P_4\rangle$; $\langle K_1+\mu_2 P_2+\mu_3 P_3, K_2+P_4, P_0+P_4, P_1\rangle$;

$\langle K_1+P_2+\alpha P_3, K_2+\beta P_3, P_0+P_4, P_1\rangle$; $\langle K_1+P_2, K_2+\alpha P_3, P_0+P_4, P_1\rangle$;

$\langle K_1+P_3, K_2+\mu P_3, P_0+P_4, P_1\rangle$; $\langle K_1, K_2+P_3, P_0+P_4, P_1\rangle$;

$\langle K_1+P_2, K_2+\beta_1 P_1+\beta_2 P_2, P_0+P_4, P_3\rangle$; $\langle K_1, K_2\pm P_2, P_0+P_4, P_3\rangle$;

$\langle K_1+P_2+\beta P_3, K_2+\delta P_3, P_0+P_4, P_1+\omega P_3\rangle$;

$\langle K_1+P_3, K_2+\mu P_3, P_0+P_4, P_1+\omega P_3\rangle$; $\langle K_1, K_2+P_3, P_0+P_4, P_1+\omega P_3\rangle$;

$\langle K_1+P_3, K_2, P_0+P_4, P_1, P_2\rangle$; $\langle K_1+P_4, K_2+\alpha P_3, P_0+P_4, P_1, P_2\rangle$;

$\langle K_1+P_2, K_2, P_0+P_4, P_1, P_3\rangle$; $\langle K_1+P_2, K_2+\alpha P_4, P_0+P_4, P_1, P_3\rangle$;

$\langle K_1, K_2+P_4, P_0+P_4, P_1, P_3\rangle$; $\langle K_1, K_2+P_3, P_0+P_4, P_1+\omega P_3, P_2\rangle$;

$\langle K_1+P_4, K_2+\mu P_3, P_0+P_4, P_1+\omega P_3, P_2\rangle$; $\langle K_1+P_3, K_2, P_0, P_1, P_2, P_4\rangle$;

$\langle K_1+P_4, K_2, P_0+P_4, P_1, P_2, P_3\rangle$; $\langle K_3, J_{04}+\alpha P_1, P_0+P_4, P_1+\omega P_3, P_2\rangle$:

$\langle K_3, J_{04}+\alpha P_2\rangle$: $\langle P_1\rangle, \langle P_0+P_4, P_1\rangle, \langle P_0+P_4, P_1+\omega P_3\rangle, \langle P_0+P_4, P_1, P_3\rangle, \langle P_0, P_1, P_3, P_4\rangle$;

$\langle K_3, J_{04}+\alpha P_3, P_0+P_4, P_1, P_2\rangle$; $\langle K_3, J_{04}+\alpha_1 P_1+\alpha_2 P_2, P_0+P_4, P_1+\omega P_3\rangle$;

$\langle K_3, J_{04}+\alpha_2 P_2+\alpha_3 P_3, P_0+P_4, P_1\rangle$;

$\langle K_3, J_{12}+cJ_{04}+\alpha P_3\rangle$: $\langle P_0+P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle (c>0)$;

$\langle K_3, J_{04}+\mu_1 P_3, J_{12}+\mu_2 P_3\rangle$: $\langle P_0+P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle (\mu_1^2+\mu_2^2>0)$;

$\langle K_1, K_2, J_{12}+\alpha P_3\rangle$; $\langle K_1, K_2, J_{12}+P_0+P_4, P_3\rangle$; $\langle K_1, K_2, J_{12}+\alpha P_3, P_0+P_4\rangle$;

$\langle K_1+P_2, K_2-P_1, J_{12}+\alpha P_3, P_0+P_4\rangle$; $\langle K_1+P_2, K_2-P_1, J_{12}, P_0+P_4, P_3\rangle$;

$\langle K_1, K_2, J_{12}+\alpha P_3, P_0+P_4, P_1, P_2\rangle$; $\langle K_1, K_2, J_{12}+\alpha P_3, P_0, P_1, P_2, P_4\rangle$;

$\langle K_1, K_2, J_{12}+P_4, P_0+P_4, P_1, P_2, P_3\rangle$;

$\langle K_1, K_2, J_{04}+\alpha P_1\rangle$: $\langle P_0+P_4, P_3\rangle, \langle P_0+P_4, P_1+\omega P_3\rangle, \langle P_0+P_4, P_1+\omega P_3, P_2\rangle$;

$\langle K_1, K_2, J_{04}+\alpha P_2\rangle$: $\langle P_0+P_4, P_1+\omega P_3\rangle, \langle P_0+P_4, P_1, P_3\rangle$;

$\langle K_1, K_2, J_{04}+\alpha P_3\rangle$: $0, \langle P_0+P_4\rangle, \langle P_0+P_4, P_1\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle$;

$\langle K_1, K_2, J_{04}+\alpha_1 P_1+\alpha_2 P_2, P_0+P_4, P_1+\omega P_3\rangle$;

$\langle K_1, K_2, J_{04}+\alpha_1 P_1+\alpha_3 P_3, P_0+P_4\rangle$; $\langle K_1, K_2, J_{04}+\alpha_2 P_2+\alpha_3 P_3, P_0+P_4, P_1\rangle$;

$\langle K_1, K_2, J_{12}+cJ_{04}+\alpha P_3\rangle$: $0, \langle P_0+P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle (c>0)$;

$\langle K_1+P_2, K_2+P_1+\beta P_2+\mu P_3, K_3+\mu P_2+\delta P_3\rangle$; $\langle K_1, K_2\pm P_2, K_3+\beta P_3\rangle$;

$\langle K_1+P_2, K_2+\beta_1 P_1+\beta_2 P_2+\alpha P_3, K_3+\delta_1 P_1+\delta_2 P_2+\delta_3 P_3, P_0+P_4\rangle$;

$\langle K_1+P_2, K_2+\beta_1 P_1+\beta_2 P_2, K_3+\alpha P_1+\delta_2 P_2+\delta_3 P_3, P_0+P_4\rangle$;

$\langle K_1+P_2, K_2+\beta_1 P_1+\beta_2 P_2, K_3+\mu P_2+\delta P_3, P_0+P_4\rangle$;

$\langle K_1, K_2\pm P_2, K_3+\beta P_3, P_0+P_4\rangle$; $\langle K_1+P_2, K_2+\alpha P_3, K_3+\beta P_2+\delta P_3, P_0+P_4, P_1\rangle$;

$\langle K_1+P_2, K_2, K_3+\mu P_2+\beta P_3, P_0+P_4, P_1\rangle$;

$\langle K_1, K_2+P_3, K_3+\beta P_2+\delta P_3, P_0+P_4, P_1\rangle$; $\langle K_1, K_2, K_3\pm P_3, P_0+P_4, P_1\rangle$;

$\langle K_1+P_3, K_2, K_3, P_0+P_4, P_1, P_2\rangle$: $\langle K_1, K_2, K_3+P_4, P_0+P_4, P_1, P_2\rangle$;

$\langle K_1+\alpha P_3, K_2, K_3+P_4, P_0+P_4, P_1, P_2\rangle$; $\langle K_1+P_4, K_2, K_3, P_0+P_4, P_1, P_2, P_3\rangle$;

$\langle K_1\pm\alpha P_1, K_2\pm\alpha P_2, J_{12}-K_3\rangle$;

$\langle K_1+\beta P_1+\mu P_2, K_2-\mu P_1+\beta P_2, J_{12}-K_3, P_0+P_4\rangle (\beta^2+\mu^2>0)$;

$\langle K_1+\alpha P_2, K_2-\alpha P_1, J_{12}-K_3, P_0+P_4, P_3\rangle$;

$\langle K_1, K_2, J_{12}-K_3+\alpha P_4, P_0+P_4, P_1, P_2, sP_3\rangle (s=0, 1)$;

$\langle J_{12}+J_{34}, J_{12}-J_{24}, J_{23}+J_{14}, J_{34}+\alpha P_0\rangle$: $0, \langle P_1, P_2, P_3, P_4\rangle$;

$\langle K_1, K_2, J_{04}+\alpha P_3, J_{12}+\mu P_3\rangle$: $0, \langle P_0+P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle$;

$\langle K_1, K_2, J_{04}, J_{12}+\alpha P_3\rangle$: $0, \langle P_0+P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle, \langle P_0, P_1, P_2, P_4\rangle$;

$\langle K_1, K_2, K_3\pm P_3, J_{12}\rangle$; $\langle K_1, K_2, K_3+\beta P_3, J_{12}+P_0+P_4\rangle$;

$\langle K_1+P_2, K_2-P_1, K_3+\beta P_3, J_{12}+\mu P_3, P_0+P_4\rangle$;

$\langle K_1, K_2, K_3\pm P_3, J_{12}+\mu P_3, P_0+P_4\rangle$; $\langle K_1, K_2, K_3, J_{12}+\alpha P_3, P_0+P_4\rangle$;

$\langle K_1+P_2, K_2-P_1, K_3, J_{12}, P_0+P_4, P_3\rangle$; $\langle K_1, K_2, K_3+P_4, J_{12}+\mu P_3, P_0+P_4, P_1, P_2\rangle$;

$\langle K_1, K_2, K_3, J_{12}+\alpha P_3, P_0+P_4, P_1, P_2\rangle$; $\langle K_1, K_2, K_3+P_4, J_{12}+\mu P_4, P_0+P_4, P_1, P_2, P_3\rangle$;

$\langle K_1, K_2, K_3, J_{12}+P_4, P_0+P_4, P_1, P_2, P_3\rangle$; $\langle K_1, K_2, K_3, J_{04}+\alpha P_1, P_0+P_4\rangle$;

$\langle K_1, K_2, K_3, J_{04}+\alpha P_2, P_0+P_4, P_1\rangle$; $\langle K_1, K_2, K_3, J_{04}+\alpha P_3, P_0+P_4, P_1, P_2\rangle$;

$\langle K_1, K_2, K_3, J_{12}+cJ_{01}+\alpha P_3\rangle$: $\langle P_0+P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle (c>0)$;

$\langle K_1, K_2, K_3, K_{04}+\mu_1 P_3, J_{12}+\mu_2 P_3\rangle$: $\langle P_0+P_4\rangle, \langle P_0+P_4, P_1, P_2\rangle (\mu_1^2+\mu_2^2>0)$.

Proof. The subalgebras of LO(1, 4) are classified by Patera *et al* (1976). For every algebra Fedorchuk (1978, 1979) has found invariant subspaces of the space *V*. Using these results together with lemmas 1-8, we will find the non-splitting subalgebras of the LP(1, 4) algebra. Below we consider some examples in detail.

Let A be a subalgebra LP(1, 4), $W = A\cap V$.

Suppose that $\pi(A) = \langle J_{12}\rangle$. Within the automorphism $\exp(t_1P_1+t_2P_2)$ the algebra A contains the element $X = J_{12}+\lambda P_0+\rho P_3+\sigma P_4 (\lambda, \rho, \sigma \in R)$. Since

$$\exp(tJ_{04})(\lambda P_0 \sigma P_4) = (\lambda \cosh t - \sigma \sinh t)P_0 + (\sigma \cosh t - \lambda \sinh t)P_4$$

then if $P_0+P_4\in W$ one can write $X = J_{12}+e^t(\lambda-\sigma)P_0+\rho P_3$. Since $\exp(\pi J_{13})(X) = -J_{12}+e^t(\lambda-\sigma)P_0-\rho P_3$, we consider $\lambda-\sigma\geqslant 0$. If $\lambda-\sigma>0$ then putting $t=-\ln(\lambda-\sigma)$, we obtain the algebra $W\supset\langle J_{12}+P_0+\rho P_3\rangle$. Applying the automorphism $\exp(tK_3)$, one can put $\rho=0$. If $\lambda-\sigma=0$ then $A = W\supset\langle J_{12}+\rho P_3\rangle$, $\rho\neq 0$.

Let $P_0+P_4\notin W$. If $P_3, P_4\in W$ then $\lambda>0$, $\rho=\sigma=0$. If $W=\langle P_4\rangle$ or $W=\langle P_1, P_2, P_4\rangle$ then $\sigma=0$. Applying the automorphism $\exp(tJ_{03})$ we reduce this case to the following ones $\lambda=\rho=1$ or $\lambda=0$, $\rho>0$.

Suppose that $\pi(A) = \langle K_1, K_2, J_{12}+cJ_{04}\rangle (c>0)$. one can suppose that A contains the elements

$$X_1 = K_1+\sum_0^4 \lambda_i P_i \qquad X_2 = K_2+\sum_0^4 \rho_i P_i \qquad X_3 = J_{12}+cJ_{04}+\sigma P_3.$$

Obviously, $[X_1, X_2] = (\lambda_2-\rho_1)(P_0+P_4)+(\lambda_0-\lambda_4)P_2-(\rho_0-\rho_4)P_1$. If $\lambda_0-\lambda_4\neq 0$ or $\rho_0-\rho_4\neq 0$ then using lemma 1, we obtain $P_1, P_2\in A$. Therefore $P_0+P_4\in A$ and one can put $\lambda_i=\rho_i=0$ for $i=0, 1, 2$. Later, $[X_3, X_1] = K_2-cK_1-c\lambda_4 P_0$, $[X_3, X_2] = -K_1-cK_2-c\rho_4 P_0$. Therefore $\lambda_3=\rho_3=0$, $\lambda_4 P_4+c\rho_4(P_4-P_0)$, $-\rho_4 P_4+c\lambda_4(P_4-P_0)\in A$. The determinant constructed by the coefficients of P_4, P_4-P_0 is equal to $c(\lambda_4^2+\rho_4^2)$. If $\lambda_4^2+\rho_4^2\neq 0$ then P_4, $P_4-P_0\in A$. So we have the algebra $\langle K_1, K_2, J_{12}+cJ_{04}+\sigma P_3, P_0+P_4, P_1, P_2, sP_0\rangle (s=0, 1)$.

Let $\lambda_0-\lambda_4=0$, $\rho_0-\rho_4=0$, $\lambda_3=\rho_3=0$. Obviously,

$$[X_3, X_1] = K_2-cK_1+\lambda_1 P_2 \lambda_2 P_1 - c\lambda_0(P_0+p_4)$$

$$[X_3, X_2] = -K_1-cK_2+\rho_1 P_2-\rho_2 P_1-c\rho_0(P_0+P_4)$$

$$[X_3, X_1]+cX_1-X_2 = (c\lambda_1-\lambda_2-\rho_1)P_1+(c\lambda_2+\lambda_1-\rho_2)P_2-\rho_0(P_0+P_4)$$

$$[X_3, X_2]+X_1+cX_2 = (\lambda_1+c\rho_1-\rho_2)P_1+(\lambda_2+c\rho_2+\rho_1)P_2+\lambda_0(P_0+P_4).$$

If on the right-hand side of one of the last two equalities some coefficients of P_1, P_2 are non-zero, so by lemmas 1 and 3 P_1, P_2, $P_0 + P_4 \in$ A. Let $c\lambda_1 - \lambda_2 - \rho_1 = 0$, $c\lambda_2 + \lambda_1 - \rho_2 = 0$, $\lambda_1 + c\rho_1 - \rho_s = 0$, $\lambda_2 + c\rho_2 + \rho_1 = 0$. The determinant formed by the coefficients of λ_1, λ_2, ρ_1, ρ_2 is equal $c^2(4+c^2)$. We obtain $\lambda_1 = \lambda_2 = 0$, $\rho_1 = \rho_2 = 0$, $\lambda_0(P_0 + P_4)$, $\rho_0(P_0 + P_4) \in$ A and therefore

$$\mathrm{A} = \mathrm{W} + \langle K_1, K_2, J_{12} + cJ_{04} + \sigma P_3 \rangle \qquad \mathrm{W} \subset \mathrm{V}.$$

Let $\pi(\mathrm{A}) = \langle J_{12}, J_{13}, J_{23}, J_{04} \rangle$. Because of the simplicity of the algebra $\langle J_{12}, J_{13}, J_{23} \rangle$ one can assume that A contains the elements J_{12}, J_{13}, J_{23}, $X = J_{04} + \Sigma\, \gamma_i P_i$ $(i = 1, 2, 3)$. Applying lemma 1 to $[J_{12}, X]$, $[J_{13}, X]$, we conclude that $\Sigma\, \gamma_i P_i \in$ A, i.e. A is a splitting algebra.

When the algebra $\pi(\mathrm{A})$ coincides with one of the following algebras: $\langle K_3, J_{04} \rangle$, $\langle K_1, K_2, J_{04} \rangle$, $\langle K_1, K_2, K_3, J_{04} \rangle$, one has to apply lemma 6. If $\pi(\mathrm{A})$ contains $J_{12} + cJ_{04}$, K_a, where $a \in I \subset \{1, 2, 3\}$, then we apply lemma 7. Thus, this theorem is proved.

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